Axial symmetry and conformal Killing vectors

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Abstract

Axisymmetric spacetimes with a conformal symmetry are studied and it is shown that, if there is no further conformal symmetry, the axial Killing vector and the conformal Killing vector must commute. As a direct consequence, in conformally stationary and axisymmetric spacetimes, no restriction is made by assuming that the axial symmetry and the conformal timelike symmetry commute. Furthermore, we prove that in axisymmetric spacetimes with another symmetry (such as stationary and axisymmetric or cylindrically symmetric spacetimes) and a conformal symmetry, the commutator of the axial Killing vector with the two others mush vanish or else the symmetry is larger than that originally considered. The results are completely general and do not depend on Einstein's equations or any particular matter content.

1 Introduction.

Symmetry is one of the important issues in Physics and, in particular, it has a remarkable relevance in General Relativity. Most of the known solutions to the Einstein field equations have been found by assuming the existence of a more or less restrictive group of isometries acting on spacetime. In fact, the classification of spacetimes themselves have been carried out with the help of the isometries they admit (sometimes in conjunction with other invariant properties such as Petrov type, etc.), which form a Lie group acting on the manifold. For a standard review of these matters see the exact solution book [1]. Most of this work has taken into account true isometries only, but not homothetic isometries and/or conformal ones. Fortunately, there has been a renewed interest for these last types of symmetries recently, and some classifications have been achieved for special matter contents in the spacetime or particular structures of the conformal Lie group, (see, for instance, [2], [3], [4] and [5]), and also some theorems relating conformal Killings with Killing tensors have been found [6].

Non isometric symmetries have a very important application in stationary and axially symmetric spacetimes, which constitute one of the outstanding scientifical subjects within General Relativity due to their importance for the description of the gravitational field of isolated objects. In fact, most *known* stationary and axisymmetric perfect-fluid solutions possess one of these symmetries. Namely, the Wahlquist solution [7] (and its limit cases found by Kramer [8], [9], [10]) have a non-trivial Killing tensor, while the solutions presented in Refs.[11][12] have a proper conformal Killing vector. It seems therefore very interesting to study general stationary and axisymmetric spacetimes with one proper (or homothetic) conformal Killing vector. This study was started by Kramer in a series of papers [13], [14], [15], where the different Bianchi types of the conformal Lie group and associated line-elements were presented. However, in these papers and also in [5] no restriction was considered for the Bianchi types and every possible case was taken into consideration. This is not what happens with Bianchi types of true isometries involving an axial Killing, as is very well known since the results of Carter [16] concerning axial symmetry appeared. Thus, for instance, it follows from Carter's paper that in stationary and axisymmetric spacetimes, the timelike and axial Killing vectors commute and, furthermore, either there are only those two symmetries or there is a larger isometry group of at least four dimensions. Do similar results hold for conformal Killings in axially symmetric spacetimes? The answer is yes, as we shall prove in this paper. More precisely, we are able to show that: 1) In axially symmetric spacetimes with one and only one conformal Killing vector, the axial Killing and the conformal Killing commute. Therefore, in axially symmetric spacetimes with a conformal Killing, if they do not commute then there is an at least three-dimensional conformal group. 2) In axially symmetric spacetimes with a *timelike* conformal Killing vector, there is no restriction in assuming that the axial Killing and the timelike conformal Killing commute and also either there are only those two symmetries or else there is a, at least, four-dimensional conformal group. 3) In axially symmetric spacetimes with one more (and only one) Killing vector and one and only one conformal Killing vector, the axial Killing commute with the other two. Therefore, we also have 4) in stationary and axially symmetric spacetimes with one conformal Killing vector, the axial Killing commutes with the two others. Thus we see that the three-dimensional conformal group of stationary and axisymmetric spacetimes with one conformal Killing cannot be arbitrary but rather it can only take one of the few forms in which the axial Killing commutes with the other two symmetries. It should be noticed that this is a general result and it does not depend on the particular form of the energy-momentum tensor or any other thing at all. Obviously, similar results hold also for cylindrically symmetric spacetimes with one conformal Killing vector.

The plan of the paper is as follows. In section 2 we define the concept of spacetime with axial symmetry and derive a series of previously known theorems in such a situation. In particular, we also recall a set of very well-known properties of the axis of symmetry and the axial Killing vector which are standard but not easily available

in the usual literature. Most of the notations and results presented there will be used repeatedly in the rest of the paper. Section 3 is devoted to presenting the main theorems in this work, which concern the combination of axial symmetry with a conformal symmetry. Namely, we prove that the axial Killing and the conformal Killing must commute if we want to have no more symmetry in spacetime. We also prove that if the conformal Killing is not normal to the axis of symmetry then either commutes with the axial Killing or there is another conformal Killing that does. A trivial corollary of this is that there is no loss of generality in assuming that a timelike conformal Killing vector commutes with the axial Killing. Of course, all these results are also valid for homothetic and true Killing vectors and thereby we reobtain, in this last case, the main results already known for stationary and axisymmetric spacetimes or cylindrically symmetric spacetimes. Finally, we devote section 4 to the case in which there is another Killing vector other than the axial Killing vector and also a proper (or homothetic) conformal Killing vector. We show that in this case the axial Killing must commute with the other symmetries. Our notation is standard and every object that appears herein is defined the first time it comes out.

2 Definitions and basic results.

A space-time, V_4, has axial symmetry if there is an effective realization of the one-dimensional torus T into V_4 that is an isometry and such that its set of fixed points constitutes a two-dimensional surface. (It can be seen that given an isometric realization of the one-dimensional torus whose fixed points form a two-dimensional surface, there is always another realization with the same properties and also effective). Mathematically, these two conditions are expressed by:

1. There is a map τ

$$\begin{aligned} \tau : T \times V_4 &\rightarrow V_4 \\ (\phi, x) &\rightarrow \tau(\phi, x) \equiv \tau_\phi(x) \end{aligned}$$

   which is a realization of the Lie group T where each τ_ϕ is an isometry of V_4.

2. The set of fixed points of τ, that is to say

$$W_2 \equiv \{ x \in V_4 \; ; \; \tau_\phi(x) = x \quad \forall \phi \in T \}$$

   is a two-dimensional surface in V_4. From now on, we shall often refer to W_2 as the *axis of symmetry*.

The Killing vector field that τ defines will be called $\vec{\sigma}$ throughout this paper. We do not assume anything about the spacelike, timelike or null character of $\vec{\sigma}$ in the spacetime,

because this character can change from one region to another. Despite this fact, it can be proven that $\vec{\sigma}$ is always spacelike in a neighbourhood of the axis W_2, as we shall presently see. At any point $Q \in W_2$ we denote by L_Q the tangent plane of W_2 at Q. Obviously, we have $L_Q \subset T_Q(V_4)$ where $T_Q(V_4)$ is the tangent space at Q. As a trivial consequence of the above, the differential map

$$\begin{array}{rcl} d\tau_\phi \mid_x : T_x(V_4) & \rightarrow & T_{\tau_\phi(x)}(V_4) \\ \vec{V} & \rightarrow & d\tau_\phi \mid_x (\vec{V}) \end{array}$$

is a linear isometry between tangent spaces, where we denote by $|_x$ the restriction to the point x. In addition, if $Q \in W_2$ then $d\tau_\phi \mid_Q$ is an isometric automorphism of $T_Q(V_4)$.

There is an easy characterization of the vectors tangent to the axis of symmetry in a fixed point, which is intuitively obvious having in mind that $d\tau_\phi$ is the linearization of the isometry group. This characterization is written down in the following Lemma (see e.g. [17]):

Lemma 1 *For points Q on the axis, $\vec{V} \in L_Q$ if and only if $d\tau_\phi \mid_Q \vec{V} = \vec{V}$ for every $\phi \in T$.*

In other words, the vectors in L_Q ($Q \in W_2$) are those invariant by $-\tau_\phi$".

For points on the axis, we define P_Q as the linear subspace of $T_Q(V_4)$ orthogonal to L_Q. Being $d\tau_\phi \mid_Q$ an isometric automorphism of $T_Q(V_4)$ it leaves P_Q invariant. The following general result about representations of the one-dimensional torus allows us to prove that the axis of symmetry is a timelike surface.

Lemma 2 *Let $\mathcal{V}$ be a two-dimensional vector space with a non-vanishing metric. If there is a (continous) representation $\{R_\phi\}$ of the Lie group T on $\mathcal{V}$ which is an isometry and also there is a $\phi_0 \in T$ such that $R_{\phi_0} \neq Id \mid_{\mathcal{V}}$, then the metric in $\mathcal{V}$ must be positive definite.*

We have already shown that P_Q is invariant under $d\tau_\phi$, which is a representation of T. If we had $d\tau_\phi \mid_{P_Q} = Id \mid_{P_Q}$ for all $\phi \in T$, then by Lemma 1 we would have $P_Q \subset L_Q$ which is impossible in a linear space with a lorentzian metric. Then, we can apply the previous Lemma to deduce that P_Q must be positive definite. Therefore L_Q is a two-dimensional lorentzian subspace and $P_Q \cap L_Q = \{\vec{0}\}$. Thus we have proven (see [16] or [17]):

Theorem 1 *The axis of symmetry, W_2, is time-oriented.*

In what follows, we are going to prove very general results concerning the commutators of general vector fields with the axial Killing vector field. These general results will be essential for the main theorems of sections 3 and 4 below. To that end, let $\vec{\alpha}$ be any vector field. The first Lemma we can show is:

Lemma 3 *For $Q \in W_2$, $\vec{\alpha}\mid_Q$ is tangent to the axis if and only if $[\vec{\alpha},\vec{\sigma}]\mid_Q = \vec{0}$.*

Proof : Due to Lemma 1 we know that $\vec{\alpha}$ is tangent to the surface of fixed points iff it is invariant by $\{\tau_\phi\}$ at points on the axis, which is equivalent to saying that $\left(\frac{d}{d\phi}d\tau_\phi\mid_{\phi=0}\right)\vec{\alpha}\mid_Q = \vec{0}$. But using the fact that in a coordinate system $\frac{d}{d\phi}(\tau_\phi)^\beta_\mu\mid_{\phi=0} = \partial_\mu\sigma^\beta$, this is equivalent to

$$\alpha^\mu\partial_\mu\sigma^\beta\mid_Q = 0.$$

Now, given that $\vec{\sigma}\mid_Q = \vec{0}$, we can rewrite this equation as $\alpha^\mu\partial_\mu\sigma^\beta - \sigma^\mu\partial_\mu\alpha^\beta\mid_Q = 0$, that is to say, $[\vec{\alpha},\vec{\sigma}]\mid_Q = \vec{0}$, and the Lemma is shown.

Thus, we have a second characterization for vectors in L_Q as those whose commutator with the axial Killing vanishes at Q. Further results of the same type can be proven, as for instance the following two Lemmas which in some sense complete the previous one.

Lemma 4 *For all vector fields $\vec{\alpha}$ and every point $Q \in W_2$, the vector $[\vec{\alpha},\vec{\sigma}]\mid_Q$ belongs to P_Q and is orthogonal to $\vec{\alpha}\mid_Q$.*

Proof : Given that $\vec{\sigma}\mid_Q = 0$ we have

$$[\vec{\alpha},\vec{\sigma}]^\beta\mid_Q = \alpha^\mu\nabla_\mu\sigma^\beta\mid_Q \qquad \forall\vec{\alpha}, \tag{1}$$

and also, as $\vec{\sigma}$ is a Killing vector,

$$[\vec{\alpha},\vec{\sigma}]^\beta\mid_Q = -\alpha^\mu\nabla^\beta\sigma_\mu\mid_Q .$$

But then $[\vec{\alpha},\vec{\sigma}]^\beta V_\beta\mid_Q = 0$ for all vectors $\vec{V} \in L_Q$, because if $\vec{V} \in L_Q$ then $V^\beta\nabla_\beta\sigma_\mu\mid_Q = 0$ as follows from Lemma 3. Therefore $[\vec{\alpha},\vec{\sigma}]\mid_Q$ is orthogonal to every vector tangent to the axis and then $[\vec{\alpha},\vec{\sigma}]\mid_Q \in P_Q$. Finally, contracting (1) with α_β it is obvious that $\vec{\alpha}\mid_Q$ and $[\vec{\alpha},\vec{\sigma}]\mid_Q$ are orthogonal to each other.

Lemma 5 *$\vec{\alpha}\mid_Q$ is not tangent to the surface of fixed points at some point $Q \in W_2$ ($\Leftrightarrow [\vec{\alpha},\vec{\sigma}]\mid_Q \neq \vec{0}$ already proven) if and only if $[\vec{\alpha},\vec{\sigma}]\mid_Q$ is linearly independent of $\vec{\alpha}\mid_Q$ and $\vec{\sigma}\mid_Q$, and obviously then $[\vec{\alpha},\vec{\sigma}]$, as a vector field, is linearly independent of $\vec{\alpha}$ and $\vec{\sigma}$.*

Proof : $[\Leftarrow]$ If $[\vec{\alpha}.\vec{\sigma}]\mid_Q$ is linearly independent of $\vec{\alpha}\mid_Q$ and $\vec{\sigma}\mid_Q$ then $[\vec{\alpha},\vec{\sigma}]\mid_Q \neq \vec{0}$ and therefore, by Lemma 3, $\vec{\alpha}$ is not tangent to the axis.

$[\Rightarrow]$ If $\vec{\alpha}\mid_Q$ is not tangent to the axis then we know by Lemmas 3 and 4 that $[\vec{\alpha},\vec{\sigma}]\mid_Q \in P_Q$, is different from zero and orthogonal to $\vec{\alpha}\mid_Q$. Given then that $[\vec{\alpha},\vec{\sigma}]\mid_Q$

is spacelike and that $\vec{\sigma}\mid_Q=\vec{0}$ it follows that $[\vec{\alpha},\vec{\sigma}]\mid_Q$ must be independent of $\vec{\alpha}\mid_Q$ and $\vec{\sigma}\mid_Q$.

The combination of the three previous Lemmas gives rise to the following interesting theorem, which will be used repeatedly as a key point in proving the main theorems of the next two sections.

Theorem 2 *Let $\vec{\alpha}$ be a vector field in an axisymmetric space-time and $Q\in W_2$.*

1. *$\vec{\alpha}\mid_Q$ is tangent to the axis at Q iff $[\vec{\alpha},\vec{\sigma}]\mid_Q=\vec{0}$.*
2. *$\vec{\alpha}\mid_Q$ $(\neq\vec{0})$ is normal to the axis at Q iff $\vec{\alpha}\mid_Q$ and $[\vec{\alpha},\vec{\sigma}]\mid_Q$ are linearly independent vectors and $[[\vec{\alpha},\vec{\sigma}],\vec{\sigma}]\mid_Q$ depends linearly on the previous.*
3. *$\vec{\alpha}$ is neither tangent nor normal to the axis at Q iff $\vec{\alpha}\mid_Q$, $[\vec{\alpha},\vec{\sigma}]\mid_Q$ and $[[\vec{\alpha},\vec{\sigma}],\vec{\sigma}]\mid_Q$ are linearly independent vectors and $[[[\vec{\alpha},\vec{\sigma}],\vec{\sigma}],\vec{\sigma}]\mid_Q$ depends linearly on the previous two.*

Proof :
Part 1) has already been shown. Let us go then to part 2). If $\vec{\alpha}\mid_Q\in P_Q$, we have seen (Lemma 4) that $[\vec{\alpha},\vec{\sigma}]\mid_Q\in P_Q$ and also that it is independent of $\vec{\alpha}\mid_Q$. Thus $\vec{\alpha}\mid_Q$ and $[\vec{\alpha},\vec{\sigma}]\mid_Q$ constitute an orthogonal basis for P_Q. But due to Lemma 4 again, $[[\vec{\alpha},\vec{\sigma}],\vec{\sigma}]\mid_Q\in P_Q$ and therefore it is a linear combination of $\vec{\alpha}\mid_Q$ and $[\vec{\alpha},\vec{\sigma}]\mid_Q$. The converse is shown in a similar manner.

Finally, with regard to part 3), if $\vec{\alpha}\mid_Q\notin P_Q$ and $\vec{\alpha}\mid_Q\notin L_Q$ then we know, due to Lemma 4, first that $[\vec{\alpha},\vec{\sigma}]\mid_Q\in P_Q$ and also that $[[\vec{\alpha},\vec{\sigma}],\vec{\sigma}]\mid_Q\in P_Q$, and second, that this last vector is orthogonal to $[\vec{\alpha},\vec{\sigma}]\mid_Q$. As $\vec{\alpha}\mid_Q\notin P_Q$, then $\vec{\alpha}\mid_Q$, $[\vec{\alpha},\vec{\sigma}]\mid_Q$ and $[[\vec{\alpha},\vec{\sigma}],\vec{\sigma}]\mid_Q$ are linearly independent vectors. Moreover, $[[[\vec{\alpha},\vec{\sigma}],\vec{\sigma}],\vec{\sigma}]\mid_Q\in P_Q$ again by Lemma 4, and then it must be a linear combination of $[\vec{\alpha},\vec{\sigma}]\mid_Q$ and $[[\vec{\alpha},\vec{\sigma}],\vec{\sigma}]\mid_Q$. The converse is now obvious.

To end this section, we recall some interesting properties concerning the intrinsic geometry of the axis of symmetry as well as the axial Killing vector and its derivatives. Most of the following properties will also be used in showing the main results of this paper presented below.

Property 1. *The axis of symmetry is an autoparallel surface.*

A surface in a manifold V is autoparallel if for every pair of vector fields defined on the surface and tangent to the surface, say $\vec{X}$ and $\vec{Y}$, the covariant derivative of $\vec{Y}$ along $\vec{X}$ remains tangent to the surface. In our case, we have, for a point $Q\in W_2$, $X^\alpha\nabla_\alpha Y^\beta\nabla_\beta\sigma^\mu\mid_Q=X^\alpha\nabla_\alpha(Y^\beta\nabla_\beta\sigma^\mu)\mid_Q-X^\alpha Y^\beta\nabla_\alpha\nabla_\beta\sigma^\mu\mid_Q$. Here the first term of the righthandside is zero because $\vec{X}$ and $\vec{Y}$ are tangent to the axis and due to Lemma 3 and formula (1). The second term in the righthandside is zero too because being $\vec{\sigma}$ a Killing vector field we can relate its second derivative with the Riemann tensor of V_4 by

$\nabla_\alpha\nabla_\beta\sigma^\mu\mid_Q=\sigma_\rho R^\rho{}_{\alpha\beta}{}^\mu\mid_Q=0$. Thus we have $X^\alpha\nabla_\alpha Y^\beta\nabla_\beta\sigma^\mu\mid_Q=0$, which implies again that $X^\alpha\nabla_\alpha Y^\beta$ is tangent to the axis and therefore that the surface W_2 is autoparallel.

Property 2. *The two second fundamental forms of the axis vanish identically.*
This is a general property of autoparallel surfaces (see e.g. [18]).

Property 3. *The tensor field $\nabla_\alpha\sigma^\rho\nabla_\beta\sigma_\rho\equiv\mathcal{H}_{\alpha\beta}$ is, at every point $Q\in W_2$, the projection tensor to P_Q.*

First of all, $\mathcal{H}_{\alpha\beta}$ is obviously a symmetric tensor. Moreover, for all vectors $\vec{V}$ tangent to the axis we have $\mathcal{H}_{\alpha\beta}V^\beta\mid_Q=0$ due to Lemma 3 and formula (1). For non-zero vector fields orthogonal to the axis, $\vec{P}\mid_Q\in P_Q$, we already know that $[\vec{P},\vec{\sigma}]\mid_Q$ belongs to P_Q , is orthogonal to $\vec{P}\mid_Q$ and different from zero. Moreover $P^\alpha\mathcal{H}_{\alpha\beta}[\vec{P},\vec{\sigma}]^\beta\mid_Q=-P^\alpha\nabla_\alpha\sigma^\rho\nabla_\rho\sigma_\beta P^\mu\nabla_\mu\sigma^\beta\mid_Q=0$, where we have used twice the skew-symmetry of $\nabla_\rho\sigma_\beta$. So we have, $P^\alpha\mathcal{H}_{\alpha\beta}\mid_Q=f(\vec{P},Q)P_\beta$ for every vector $\vec{P}\mid_Q\in P_Q$, where f is, in principle, a function depending on the point of the axis and the vector $\vec{P}\mid_Q$.

Let us now take another vector $\vec{R}\in P_Q$. We find $P^\alpha\mathcal{H}_{\alpha\beta}\mid_Q R^\beta=f(\vec{P},Q)P_\beta R^\beta\mid_Q=f(\vec{R},Q)P_\beta R^\beta\mid_Q$ due to the symmetry of $\mathcal{H}_{\alpha\beta}\mid_Q$ and then f is a function depending only on the point in the surface. Taking two vectors fields on W_2, $\vec{u}$ and $\vec{v}$, which are tangent to the axis and such that $u^\alpha u_\alpha=-1,\ v^\alpha v_\alpha=1$ and $u^\alpha v_\alpha=0$ everywhere on W_2, we can write

$$\mathcal{H}_{\alpha\beta}\mid_Q=f(Q)(g_{\alpha\beta}+u_\alpha u_\beta-v_\alpha v_\beta)\mid_Q$$

where g is the metric in V_4. From the relation $\nabla_\alpha\nabla_\beta\sigma^\mu\mid_Q=0$ it is immediate to deduce that $\nabla_\lambda\mathcal{H}_{\alpha\beta}\mid_{W_2}=0$ and from this expression one can easily find that f is in fact a constant rather than a function on W_2. It only remains to prove that this constant is equal to one. For a vector field such that $\vec{P}\mid_Q(\neq 0)\in P_Q$ we have

$$P^\alpha\mathcal{H}_{\alpha\beta}P^\beta\mid_Q=P^\alpha\nabla_\alpha\sigma^\rho\nabla_\beta\sigma_\rho P^\beta\mid_Q=[\vec{P},\vec{\sigma}]^\rho[\vec{P},\vec{\sigma}]_\rho\mid_Q=fP^\rho P_\rho\mid_Q$$

and then f is a positive constant, say $f=a^2$.

It is a known result in the theory of transformation groups (see e.g. [18]) that

$$d\tau_\phi(\vec{\alpha})=\vec{\alpha}+\phi[\vec{\alpha},\vec{\sigma}]+\frac{\phi^2}{2}[[\vec{\alpha},\vec{\sigma}],\vec{\sigma}]+\dots$$

In our case we have for the vector field $\vec{P}$, $[[\vec{P},\vec{\sigma}],\vec{\sigma}]^\beta\mid_Q=P^\alpha\nabla_\alpha\sigma^\rho\nabla_\rho\sigma^\beta\mid_Q=-P^\alpha\mathcal{H}^\beta_\alpha\mid_Q=-a^2P^\beta\mid_Q$, and therefore it follows that, at $Q\in W_2$,

$$d\tau_\phi\mid_Q(\vec{P})=\vec{P}\mid_Q+\phi[\vec{P},\vec{\sigma}]\mid_Q-\frac{\phi^2}{2}a^2\vec{P}\mid_Q-\frac{\phi^3}{6}a^2[\vec{P},\vec{\sigma}]\mid_Q+\dots=$$

$$=\cos(a\phi)\vec{P}\mid_Q+\frac{1}{a}\sin(a\phi)[\vec{P},\vec{\sigma}]\mid_Q$$

But if we now choose the standard parametrization for the torus such that ϕ goes from 0 to 2π, then we have that $d\tau_{2\pi}\mid_Q = Id\mid_Q$. In consequence, and given that the realization is effective, we have $a=1$ as required.

Let us note finally that from the definition of $\mathcal{H}_{\alpha\beta}\mid_Q$ we can easily find the expression

$$\mathcal{H}_{\alpha\beta}\mid_Q = \frac{1}{2}\nabla_\alpha\nabla_\beta(\vec{\sigma}^2)\mid_Q . \tag{2}$$

Property 4. *For any point $Q \in W_2$ there is a neighbourhood of Q such that $\vec{\sigma}^2$ is non-negative and it is zero only at points on the axis.*

This is trivial from the expression (2) just found in the previous proof because $\mathcal{H}_{\alpha\beta}\mid_Q$ is positive definite and also both $\vec{\sigma}^2$ and $\nabla_\rho(\vec{\sigma}^2)$ vanish at the axis. Therefore, there always exists a neighbourhood of the axis where the axial Killing vector field is spacelike.

Property 5. *At the axis of symmetry we have*

$$\frac{\nabla_\rho(\vec{\sigma}^2)\nabla^\rho(\vec{\sigma}^2)}{4\vec{\sigma}^2} \longrightarrow 1.$$

This is the popular *regularity condition* of the axis of symmetry, which usually is assumed to assure the standard 2π-periodicity of ϕ. We will prove this result by using the projector $\mathcal{H}_{\alpha\beta}$ constructed above. Let us evaluate $\nabla_\alpha\vec{\sigma}^2\nabla^\alpha\vec{\sigma}^2 = 4\sigma_\rho\sigma_\nu\nabla_\alpha\sigma^\nu\nabla^\alpha\sigma^\rho = 4\sigma^\rho\sigma^\nu\mathcal{H}_{\rho\nu}$ near the axis up to second order, by means of an expansion in a coordinate system. At any point with coordinates x^ν, in some neighbourhood of $Q \in W_2$ with coordinates Q^ν, we obviously have

$$\sigma^\mu(x) = (x^\nu - Q^\nu)\nabla_\nu\sigma^\mu\mid_Q + o(2)$$

$$\mathcal{H}_{\mu\nu}(x) = \mathcal{H}_{\mu\nu}\mid_Q + o(1)$$

and then we find $\sigma^\mu\sigma^\nu\mathcal{H}_{\mu\nu}(x) = (x^\alpha - Q^\alpha)\nabla_\alpha\sigma^\mu(x^\beta - Q^\beta)\nabla_\beta\sigma^\nu\nabla_\mu\sigma^\rho\nabla_\nu\sigma_\rho\mid_Q + o(3) = (x^\alpha - Q^\alpha)(x^\beta - Q^\beta)\mathcal{H}_\alpha{}^\rho\mathcal{H}_{\beta\rho} + o(3) = (x^\alpha - Q^\alpha)(x^\beta - Q^\beta)\mathcal{H}_{\alpha\beta}\mid_Q + o(3)$.
On the other hand from the expansion of $\sigma^\mu(x)$ we have

$$\begin{aligned}\vec{\sigma}^2(x) = (x^\alpha - Q^\alpha)(x^\beta - Q^\beta)\nabla_\alpha\sigma_\rho\nabla_\beta\sigma^\rho\mid_Q + o(3) = \\ (x^\alpha - Q^\alpha)(x^\beta - Q^\beta)\mathcal{H}_{\alpha\beta}\mid_Q + o(3)\end{aligned}$$

From this expression follows immediatly the property above. Moreover, in a linearized point of view of the manifold, and in a neighbourhood of $Q \in W_2$, we can consider $(x^\alpha - Q^\alpha)$ as a vector pointing from Q to x. Its double contraction with $\mathcal{H}_{\alpha\beta}$ gives the modulus of the normal component of this vector in P_Q or, in other words, the distance from the axis to the point x. The last expression shows that the norm $\vec{\sigma}^2\mid_x$ coincides, at first non-trivial order, with this distance from x to the axis of symmetry. If we

now consider the orbit of the axial symmetry which passes through x we have that its length is $2\pi\vec{\sigma}^2 \mid_x$, because the norm of a Killing vector field remains constant alongs its orbits. Then we have found that the length of the axial orbit is, at first relevant order, 2π times the distance from x to the axis of symmetry, which states what is ususally called *elementary flatness*.

3 Axially symmetric spacetimes with conformal symmetries.

The results of the previous section hold in space-times with axial symmetry. However, we are interested in V_4's which also have conformal symmetries. This section is devoted to presenting the basic results appearing when these two symmetries hold simultaneously, which are the main results in this paper.

Let us start by recalling that a conformal isometry of V_4 is a diffeomorphism

$$\begin{array}{rcl} L : V_4 & \rightarrow & V_4 \\ x & \rightarrow & L(x) \end{array}$$

such that $L^\star g = e^U g$, where $L^\star$ is the pullback of L, g is the metric tensor field in V_4 and e^U is a function called the conformal factor of L. We consider the case where there is a one-dimensional group of transformations in V_4, say "Λ_s", which are conformal isometries with associated conformal Killing vector field $\vec{\lambda}$. That is to say, there is a map

$$\begin{array}{rcl} \Lambda : \Re \ (or\ T) \ \times V_4 & \rightarrow & V_4 \\ (s,x) & \rightarrow & \Lambda(s,x) \equiv \Lambda_s(x) \end{array}$$

which is a realization of the one-dimensional Lie group $\Re$ (or T) in V_4, where each Λ_s is a conformal isometry of the spacetime:

$$\Lambda_s^\star g = e^{U_s} g. \tag{3}$$

The infinitesimal generator of this group of transformations, $\vec{\lambda}$, satisfies then

$$\mathcal{L}_{\vec{\lambda}} g = \Psi g \tag{4}$$

where $\mathcal{L}_{\vec{\lambda}}$ is the Lie derivative with respect to $\vec{\lambda}$ and $\Psi(x)$ is a function called the scale factor of $\vec{\lambda}$. It is easily verified that

$$U_s(x) = \int_0^s (\Psi \circ \Lambda_h(x)) dh \Longleftrightarrow \Psi(x) = \frac{d}{ds} U_s(x) \mid_{s=0} . \tag{5}$$

We begin by proving the following general result that we will need later:

Proposition 1 *Let L be a conformal transformation in V_4 with conformal factor e^U and $\vec{\xi}$ a Killing vector field. The vector field defined by*

$$\vec{\zeta}(y) = dL(\vec{\xi}(x)) \quad y = L(x)$$

is a conformal Killing field and it is a Killing field if and only if $\vec{\xi}(U) = 0$.

Proof : We have $L^\star g = e^U g$ where U is a scalar function, and $\Xi_s^\star g = g$ where $\{\Xi_s\}$ is the local group of transformations generated by $\vec{\xi}$.

Of course L^{-1} is a conformal transformation and it is easy to check that

$$L^{-1\star}(g) = e^{-(U \circ L^{-1})} g$$

Let us now define $L_s = L \circ \Xi_s \circ L^{-1}$. Obviously $\{L_s\}$ is a local one-parameter group of conformal isometries. In fact

$$L_s^\star g \mid_x = (L \circ \Xi_s \circ L^{-1})^\star(g) \mid_x = (L^{-1})^\star \circ (\Xi_s)^\star \circ (L)^\star(g) \mid_x = (L^{-1^\star} \circ \ \Xi_s{}^\star)(e^U g) \mid_x$$

$$= L^{-1\star}(e^{U \circ \Xi_s} g) \mid_x = e^{U \circ \Xi_s \circ L^{-1}} e^{-U \circ L^{-1}} g \mid_x = e^{U \circ \Xi_s \circ L^{-1} - U \circ L^{-1}} g \mid_x .$$

A very well-known standard fact (see, for example [19]) is that the infinitesimal generator of $\{L_s\}$ is given by $\vec{\zeta} = dL(\vec{\xi})$. Thus, we have that $dL(\vec{\xi})$ is a conformal Killing vector field. It will actually be a Killing vector field if and only if, for every value of s $U \circ \Xi_s \circ L^{-1} - U \circ L^{-1} = 0$, that is to say, iff $U(\Xi_s(x)) = U(x)$ which means that U is constant along the orbits of $\vec{\xi}$. So $dL(\vec{\xi})$ is a Killing vector field if and only if $\vec{\xi}(U) = 0$, as we wanted to prove.

The same result can be essentially deduced by using the expression

$$\mathcal{L}_{\vec{\xi}}(L^\star g) = L^\star(\mathcal{L}_{dL(\vec{\xi})} g)$$

that can be shown for all vectors $\vec{\xi}$, all covariant tensors g and all $L : V_4 \rightarrow V_4$.

Corollary *Let $\{\Lambda_s\}$ be a one-dimensional group of conformal transformations in V_4 with scale factor Ψ and $\vec{\xi}$ a Killing vector field. For each s, the vector field defined by*

$$\vec{\zeta}_s(y) = d\Lambda_s(\vec{\xi}(x)) \quad y = \Lambda_s(x)$$

is a conformal Killing vector field and it is a Killing vector field iff $\vec{\xi}(\Psi) = 0$.

The corollary follows because from the previous Lemma the condition is $\vec{\xi}(U_s) = 0$ and then from formula (5) this is equivalent to $\vec{\xi}(\Psi) = 0$.

We can now come back to the case in which the spacetime is axisymmetric. The notations are then those of section 2. The first interesting fact is:

Proposition 2 *In an axisymmetric spacetime, let $\vec{\lambda}$ be a conformal Killing vector field tangent to the axis for all $Q \in W_2$ and with associated scale factor Ψ. Then $[\vec{\lambda}, \vec{\sigma}] = \vec{0}$ if and only if $\vec{\sigma}(\psi) = 0$.*

Proof : For every s we define the following realization of T into V_4

$$\begin{array}{rcl} \tilde{\tau}^{(s)} : T \times V_4 & \rightarrow & V_4 \\ (\phi, x) & \rightarrow & \tilde{\tau}^{(s)}(\phi, x) \equiv \tilde{\tau}^{(s)}_{\phi}(x) \end{array}$$

by $\tilde{\tau}^{(s)}_{\phi} = \Lambda_s \circ \tau_\phi \circ \Lambda_{-s}$ where $\{\Lambda_s\}$ is the local one-parameter group of transformations generated by $\vec{\lambda}$. Given that $\vec{\lambda}$ is tangent to the axis, it is straightforward to see that the fixed points of $\tilde{\tau}^{(s)}_{\phi}$ are the fixed points of τ_ϕ, and viceversa. We start by proving first the converse implication of the Proposition.

[$\Leftarrow$] If $\vec{\sigma}(\Psi) = 0$ then $\tilde{\tau}^{(s)}_{\phi}$ is an isometry for each ϕ (because of Proposition 1 and its corollary) which is a realization of the Lie group T and with the same axis of symmetry that τ. A Lemma due to Carter [16] implies that they are the same group of transformations. So we have for every s and ϕ, $\Lambda_s \circ \tau_\phi \circ \Lambda_{-s} = \tau_\phi$, which implies (see e.g. [19]) $[\vec{\lambda}, \vec{\sigma}] = \vec{0}$.

[$\Rightarrow$] If $[\vec{\lambda}, \vec{\sigma}] = \vec{0}$, then we have [19] $\Lambda_s \circ \tau_\phi \circ \Lambda_{-s} = \tau_\phi$ which means that $\tilde{\tau}^{(s)}_{\phi}$ is an isometry. It follows from Lemma 6 and its corollary that $\vec{\sigma}(\Psi) = 0$.

Therefore, the necessary and sufficient condition such that a conformal Killing vector field tangent to the axis commutes with the axial Killing is that the scale factor be constant along the orbits of the axial Killing vector field . A trivial consequence is that all homothetic Killing vector fields (and also all Killing vector fields) tangent to the axis commute with the axial symmetry. Despite of this, it seems that, in principle, this is not true for general conformal Killing vector fields. We shall see, however, that this property does hold for general conformal Killings. In order to prove it, we first need the following fundamental result, which strengthes previous similar results [16] and states that an axial conformal Killing vector field and an axial Killing vector field with the same axis of symmetry in a given spacetime must coincide.

Theorem 3 *Let $\{\eta_\phi\}$ be an effective realization of the Lie group T that is a conformal isometry with a surface of fixed points $\tilde{W}_2$ in an axisymmetric space-time. If $\tilde{W}_2 = W_2$, then $\{\eta_\phi\}$ is in fact an isometry and coincides with the axial isometry.*

Proof : We call $\vec{\mu}$ the infinitesimal generator of $\{\eta_\phi\}$. The method of the proof is to see that both $\vec{\mu}$ and $\vec{\sigma}$ satisfy the same equations and the same initial conditions, so that they must be identical. The vector field $\vec{\mu}$ is a conformal Killing and then it satisfies

$$\mathcal{L}_{\vec{\mu}}(g) = \tilde{\Psi} g \tag{6}$$

where $\tilde{\Psi}$ is the associated scale factor. On the other hand, it follows from (5) that for conformal Killing vector fields the finite transformation verifies

$$\eta_\phi^\star(g)(x) = \exp\left(\int_0^\phi (\tilde{\Psi}\circ\eta_\rho)(x)d\rho\right)\ g(x).$$

Let Q be a point on the axis of symmetry W_2 of $-\eta_\phi$". As $\eta_{2\pi} = Id\mid_{V_4}$, we then have $\int_0^{2\pi}(\tilde{\Psi}\circ\eta_\rho)(Q)d\rho = 0$. But Q is a fixed point and thus, for every $\rho\in T$, $\eta_\rho(Q) = Q$ so that from the previous equation we obtain that the scale factor vanishes at the axis:

$$\tilde{\Psi}\mid_{W_2} = 0. \tag{7}$$

Analogously to Lemma 1, the vectors tangent to the axis of symmetry in a fixed point can be characterized by

$$\vec{V}\in L_Q \Leftrightarrow d\eta_\phi\mid_Q(\vec{V}) = \vec{V}\quad \forall\phi\in T$$

Our aim, now, is to see that for a point $Q\in W_2$, the differential maps $d\tau_\phi\mid_Q$ and $d\eta_\phi\mid_Q$ coincide. First, $d\eta_\phi\mid_Q$ is an automorphism of $T_Q(V_4)$ and leaves L_Q invariant. Moreover it is an isometry of $T_Q(V_4)$ because $\tilde{\Psi}(Q) = 0$. In consequence P_Q, the complement orthogonal of L_Q in $T_Q(V_4)$, is an invariant subspace of $d\eta_\phi\mid_Q$. Thus, we have two effective isometric realizations of the Lie group T on P_Q , which is a two-dimensional vector space with positive definite metric, and then they must coincide. So we have for every point Q on the axis $d\tau_\phi\mid_Q = d\eta_\phi\mid_Q$, from where it can be immediately deduced that

$$\nabla\vec{\mu}\mid_Q = \nabla\vec{\sigma}\mid_Q \tag{8}$$

where we have used

$$\vec{\sigma}\mid_Q = \vec{\mu}\mid_Q = \vec{0}. \tag{9}$$

From formula (8) it follows that for every vector $\vec{V}\in L_Q$ we have

$$V^\alpha\nabla_\alpha(\nabla_\beta\mu_\gamma)\mid_Q = V^\alpha\nabla_\alpha(\nabla_\beta\sigma_\gamma)\mid_Q \tag{10}$$

But, as $\vec{\mu}$ is a conformal Killing vector field and $\vec{\sigma}$ is a Killing vector field, they obey the following relations, see e.g. [20]

$$\begin{aligned}\nabla_\rho\nabla_\nu\mu_\alpha &= \mu_\delta R^\delta_{\rho\nu\alpha} + \frac{1}{2}(g_{\nu\alpha}\nabla_\rho\tilde{\Psi} + g_{\alpha\rho}\nabla_\nu\tilde{\Psi} - g_{\rho\nu}\nabla_\alpha\tilde{\Psi}), \qquad (11)\\ \nabla_\rho\nabla_\nu\sigma_\alpha &= \sigma_\delta R^\delta_{\rho\nu\alpha}\end{aligned}$$

and this together with (9) gives

$$\nabla_\rho\nabla_\nu\mu_\alpha\mid_Q = \frac{1}{2}(g_{\nu\alpha}\nabla_\rho\tilde{\Psi} + g_{\alpha\rho}\nabla_\nu\tilde{\Psi} - g_{\rho\nu}\nabla_\alpha\tilde{\Psi})\mid_Q, \tag{12}$$

$$\nabla_\rho\nabla_\nu\sigma_\alpha\mid_Q = 0. \tag{13}$$

Combining (10) with (13) we can write, for every $\vec{V} \in L_Q$

$$V^\alpha \nabla_\alpha (\nabla_\beta \mu_\gamma) \mid_Q = V^\alpha \nabla_\alpha (\nabla_\beta \sigma_\gamma) \mid_Q = 0$$

and this last expression together with (12) leads us to

$$\frac{1}{2}(V_\alpha \nabla_\nu \tilde{\Psi} - V_\nu \nabla_\alpha \tilde{\Psi}) \mid_Q = 0 \tag{14}$$

where we have used (see formula (5)) that $V^\rho \nabla_\rho \tilde{\Psi} \mid_Q = 0$, that is to say, $\nabla_\alpha \tilde{\Psi} \mid_Q \in P_Q$. But then, being $\vec{V} \in L_Q$ and $\nabla_\alpha \tilde{\Psi} \in P_Q$, expression (14) implies necessarily

$$\nabla_\alpha \tilde{\Psi} \mid_Q = 0. \tag{15}$$

Equations (6) for a conformal Killing vector field $\vec{\mu}$ with scale factor $\tilde{\Psi}$ are

$$\nabla_\rho \mu_\delta + \nabla_\delta \mu_\rho = \tilde{\Psi} \ g_{\rho\delta}. \tag{16}$$

These equations are not written in normal form, but well-known consequences of them [20] are expression (11) and the following

$$\nabla_\beta \nabla_\alpha \tilde{\Psi} = \frac{1}{6}(\mathcal{L}_{\vec{\mu}} R + \tilde{\Psi} R) g_{\beta\alpha} - \mathcal{L}_{\vec{\mu}} R_{\beta\alpha}, \tag{17}$$

where $R_{\beta\alpha}$ and R are the Ricci tensor and the scalar curvature of the spacetime, respectively.

Expressions (16), (11) and (17) imply, among other things, that if at some point $x \in V_4$ a conformal Killing vector field and its scale factor have the following properties

$$\vec{\mu} \mid_x = \vec{0} \quad , \ \nabla_\nu \mu_\rho \mid_x = 0$$
$$\tilde{\Psi}(x) = 0 \quad , \ \nabla_\nu \tilde{\Psi} \mid_x = 0$$

then the conformal Killing vector field must be zero everywhere.

In our case, the vector field $\vec{\mu} - \vec{\sigma}$ is a conformal Killing with scale factor $\tilde{\Psi}$ and with the properties written above, because of formulae (5), (15), (9) and (8). Consequently, $\vec{\mu} - \vec{\sigma}$ must be zero everywhere. Therefore, $\vec{\mu}$ is a Killing vector field and the conformal symmetry it defines is, in fact, the axial symmetry we already had in the spacetime.

We are now prepared to prove the main theorems in this paper. Let us start with the following important result

Theorem 4 *In an axially symmetric spacetime, if $\vec{\lambda}$ is a conformal Killing vector field tangent to the axis of symmetry for all $Q \in W_2$, then*

$$[\vec{\lambda}, \vec{\sigma}] = \vec{0}$$

and also Proposition 2 implies that $\vec{\sigma}(\Psi) = 0$, where Ψ is the scale factor associated with $\vec{\lambda}$.

Proof : Let us call, as before, $\{\Lambda_s\}$ the local group of transformations generated by $\vec{\lambda}$ with s in some neighbourhood of zero and $\tilde{\tau}_\phi^{(s)} = \Lambda_s \circ \tau_\phi \circ \Lambda_{-s}$. We have already shown (proof of Proposition 2) that the set of fixed points of $\tilde{\tau}^{(s)}$ are the same that the set of fixed points of τ. Then we can apply the previuos theorem to see that, for every ϕ and s,

$$\tilde{\tau}_\phi^{(s)} = \Lambda_s \circ \tau_\phi \circ \Lambda_{-s} = \tau_\phi$$

and consequently, we get

$$[\vec{\lambda}, \vec{\sigma}] = \vec{0}.$$

This result allows us to prove that in axially symmetric spacetimes with a conformal Killing and no other conformal Killing (nor Killing) vector fields, the axial symmetry and the conformal symmetry commute. More precisely we have the following theorem.

Theorem 5 *In an axisymmetric spacetime with a conformal Killing vector $\vec{\lambda}$, if there is no more conformal symmetry then*

$$[\vec{\lambda}, \vec{\sigma}] = \vec{0}.$$

Proof : For a point $Q \in W_2$, if we had $[\vec{\sigma}, \vec{\lambda}] \mid_Q \neq \vec{0}$ then (Theorem 2) $[\vec{\sigma}, \vec{\lambda}] \mid_Q$ would be linearly independent of $\vec{\lambda} \mid_Q$ and $\vec{\sigma} \mid_Q$, and then as vector fields they are independent. As $[\vec{\sigma}, \vec{\lambda}]$ is a conformal Killing vector field we would have more symmetry against the hypothesis. So, for every $Q \in W_2$, $[\vec{\sigma}, \vec{\lambda}] \mid_Q = \vec{0}$ and then, because of the same Theorem 2, $\vec{\lambda}$ is tangent to the surface of fixed points for all $Q \in W_2$. Theorem 4 implies then that $[\vec{\sigma}, \vec{\lambda}] = \vec{0}$.

Another immediate consequence of the previous results is that, in an axially symmetric spacetime, there is no restriction in assuming that a timelike conformal Killing commutes with the axial Killing (this was already known for Killing vector fields, see [16]). The precise statement is given in the corollary following the next Proposition.

Proposition 3 *In an axisymmetric spacetime, let $\vec{\lambda}$ be a conformal Killing vector field which does not commute with $\vec{\sigma}$. If at some point Q of the axis $\vec{\lambda} \mid_Q$ is not normal to the surface of fixed points, then there always exists another conformal Killing vector field that commutes with the axial Killing vector field.*

Proof : We know that $\vec{\lambda} + [[\vec{\lambda}, \vec{\sigma}], \vec{\sigma}]$ is a conformal Killing vector field. We are going to see that this vector field is tangent to the axis everywhere on the axis. In fact, at any point $Q \in W_2$, we can decompose $\vec{\lambda} \mid_Q$ in an unique way into its components tangent and normal to the surface: $\vec{\lambda} \mid_Q = \vec{\lambda}_\parallel \mid_Q + \vec{\lambda}_\perp \mid_Q$. On the other hand, we have

$[[\vec{\lambda},\vec{\sigma}],\vec{\sigma}]\mid_Q = -\vec{\lambda}_\perp \mid_Q$ due to property 3 of axial symmetry listed at the end of section 2. In consequence, the conformal Killing vector field $\vec{\lambda}+[[\vec{\lambda},\vec{\sigma}],\vec{\sigma}]$ is tangent to the axis at every point $Q \in W_2$ and then, by Theorem 4, it commutes with the axial Killing vector field. If $\vec{\lambda}$ is not orthogonal to the axis at some point $Q \in W_2$ then the conformal Killing vector field considered is not identically vanishing.

Corollary *In an axisymmetric spacetime, if there is a timelike conformal Killing field, then there always exists a timelike conformal Killing field that commutes with the axial Killing field.*

The corollary is evident because a timelike vector field cannot be orthogonal to the axis anywhere and its tangent component is obviously timelike because of Theorem 1. Thus, the derived conformal Killing field of the previous proposition commutes with tha axial symmetry and is also timelike at least in the region where the original one was timelike, as can be checked.

Let us remark that all the results shown in this section for conformal Killings hold also for homothetic Killings and real Killings, as is obvious. Most of these results were known for Killing fields but, as far as we know, they were previously unknown for general conformal Killing vector fields.

4 Axisymmetric spacetimes with another symmetry and a conformal symmetry.

Until now we have been considering an axisymmetric spacetime with a conformal Killing vector field. In General Relativity it has much interest the case of axisymmetric spacetimes with another symmetry which commutes with the axial symmetry, for example stationary and axisymmetric spacetimes or cylindrically symmetric spacetimes. It is obvious that all we have done in the case of conformal Killing fields applies for Killing fields as well, and so we can recover the main results proved by Carter in the early seventies. Moreover, it has been recently found [13] that a class of stationary and axisymmetric exact solutions [11] possesses a conformal Killing vector and a new family of stationary and axisymmetric exact solution with the same property has been constructed [12]. Due to the importance that these type of metrics may have in describing the gravitational field of isolated objects, as explained in the Introduction, some papers have recently appeared in the literature considering the case of stationary and axisymmetric exact solutions with a third proper conformal Killing vector field and studying the different Bianchi types that these three vector fields can adopt.

In that direction the previous Lemmas and Theorems allow us to show the following result.

Lemma 6 *In an axisymmetric spacetime with another Killing vector field $\vec{\xi}$ and a*

proper (or homothetic) conformal Killing field $\vec{\lambda}$, if there is no more conformal symmetry then

$$[\vec{\sigma}, \vec{\xi}] = \vec{0}$$

and also $\vec{\lambda}\mid_Q$ is tangent to the axis for all $Q \in W_2$.

Here no more conformal symmetry means no more Killing vector fields and no more conformal Killing vector fields either.

Proof : The first part of this proof follows exactly the same steps that those in the proof of Theorem 5, because the results proven for conformal Killing vector fields are also true for Killing vector fields and we are assuming that there are no Killing fields other than $\vec{\sigma}$ and $\vec{\xi}$.

With respect to the second part, we know that $[\vec{\lambda}, \vec{\sigma}]$ is a conformal Killing vector field, so that the only possibility for not having more conformal symmetry is that

$$[\vec{\lambda}, \vec{\sigma}] = a\vec{\sigma} + b\vec{\lambda} + c\vec{\xi} \tag{18}$$

where a, b and c are constants. If we commute now with $\vec{\sigma}$ we obtain $[[\vec{\lambda}, \vec{\sigma}], \vec{\sigma}] = b[\vec{\lambda}, \vec{\sigma}]$. But at points of the axis we know by Lemma 4 that $[[\vec{\lambda}, \vec{\sigma}], \vec{\sigma}]\mid_{W_2}$ and $[\vec{\lambda}, \vec{\sigma}]\mid_{W_2}$ are normal to the axis and orthogonal to each other. It must be then, $b[\vec{\lambda}, \vec{\sigma}]\mid_{W_2} = \vec{0}$. If $[\vec{\lambda}, \vec{\sigma}]\mid_{W_2} = \vec{0}$ we are done because of Lemma 3. Otherwise we should have $b = 0$, and then from (18) and given that $\vec{\xi}\mid_{W_2}$ is tangent to the axis we obtain again by Lemma 4 that $[\vec{\lambda}, \vec{\sigma}]\mid_{W_2} = \vec{0}$, and using Lemma 3, $\vec{\lambda}\mid_{W_2}$ is tangent to the axis.

Trivial consequence of this Lemma and Theorem 4 is the following important result.

Theorem 6 *In an axisymmetric spacetime with another Killing vector field $\vec{\xi}$ and a conformal Killing vector field $\vec{\lambda}$, if there is no more conformal symmetry then*

$$[\vec{\sigma}, \vec{\xi}] = \vec{0},$$

$$[\vec{\sigma}, \vec{\lambda}] = \vec{0}.$$

We see, therefore, that in stationary and axisymmetric spacetimes, if there is one (and only one) proper (or homothetic) conformal Killing vector field, then it must commute with the axial symmetry. Of course, the same happens in cylindrically symmetric spacetimes. This is a very interesting result and, in fact, it simplifies largely the Bianchi types one has to study in these cases. Thus, for instance, it has been recently considered the case of stationary and axisymmetric perfect-fluid spacetimes in Refs.[15], [5]. In these papers, Bianchi types with $[\vec{\sigma}, \vec{\lambda}] \neq \vec{0}$ have been studied with the result of the impossibility of getting solutions to the field equations for a perfect-fluid energy-momentum tensor. In fact, as we have shown, it does not matter what the

energy-momentum is, there exists *no* spacetime with that property. Theorem 6 above should be taken into account for future work in spacetimes with two symmetries and one conformal symmetry, whenever one of the symmetries is required to be axial. Similarly, for cases with axial symmetry and only one more conformal symmetry we have proven in the previous section that these two symmetries must commute. Therefore, if we want to study conformally stationary and axially symmetric spacetimes, we can assume without restriction that these two symmetries commute (analogously to what happens in stationary and axisymmetric manifolds), and set up the coordinate system accordingly. Many other consequences can be extracted from the results herein shown, but as they are self-evident we do not believe necessary to explain them here further.

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